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STUDY OF MOST CELEBRATED BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION

Akanksha S.Shinde*

ABSTRACT

The Black-Scholes partial differential equation is used for valuing European or American put and call option.It is the one of the most effective way for pricing options. The aim of this paper is to study the development and derivation of Black-Scholes partial differential equation. We discuss some useful definitions and derivations which are useful in the development of Black-Scholes Partial differential equation.

Keywords:Black-Scholes partial differential equation, Ito's Lemma, Put Option, Risk-less interest rate, Strike price.

* **Department of Mathematics,VPM's B. N. Bandodkar College of Science Thane, Maharashtra, India**

1. INTRODUCTION

Currently the study of partial differential equations is one of the fields within the Applied Mathematics that exhibits increasing interest because they allow to solve a wide variety of problems from science and engineering. The main topic in Financial Mathematics is concerned with Black-Scholes partial differential equation which involves financial processes such as stock prices, interest rates, exchange rates and pricing derivatives on basis underlying asset. Study of Black-Scholes partial differential equation is one of such kind [2,3]. This equation has been increasingly attracting interest over the last two decades since it provides effectively the values of option. In the year 1973, Fischer Black and Myron Scholes develop the original option pricing formula and it is published in the paper entitled "The Pricing of Options and Corporate Liabilities" in the journal of political economy [1]. The black-Scholes differential equation governs the price of the option over time. Due to its simplicity and clarity to obtain the price option calls, the Black-Scholes partial differential equation is used in financial engineering very often [4,5].

In our study, we discuss some basic definitions, derivations and lemma's which are useful in the development of Black-Scholes partial differential equation. Also we study Ito' Lemma from Ito's Calculus.Also, we derive Black-Scholes partial differential equation.

We organize the paper as follows: The section 2, is devoted for some definitions and derivations. In section 3, we study Ito's lemma which is used to derive the Black-Scholes differential equation. Last section is devoted to the derivation of Black-Scholes partial differential equation.

2. DEFINITIONS IN FINANCIAL MATHEMATICS

2.1: Option

A security giving the right to buy or sell an asset, subject to certain conditions, within a specified period of time is called as an option [4].

There are two types of options which are defined as follows:

(i) **Call Option**: An option which grants its holder the right to buy the underlying asset at a strike price at some moment in the future is called as call option.

(ii) **Put Option**: An option which grants its holder the right to sell the underlying asset at a strike price at some moment in the future is called as put option.

2.2: Expiration Date

The date on which an option right or warrant expires, and becomes worthless if not exercised is called an expiration date. There are two different types of options with respect to expiration.

(i) **European Option**: An option which cannot be exercised until the expiration date is called an European Option.

(ii) **American Option**: An option which can be exercised at any time up to and including the expiration date is called as an American Option.

2.3: Risk-Less Interest Rate

The annual interest rate of bonds or other "risk-free" investments, is called as the risk-less interest rate. It is denoted by r.

2.4: Volatility

A measure for variation of price of a financial instrument over time is called volatility (Baxter, 1996). There are two important types of volatility as follows:

(i) **Implied Volatility**: Volatility derived from the market price of a market traded derivative is called implied volatility.

(ii) **Historic Volatility**: Volatility derived from time series of past market prices is called historic volatility.

2.5: Strike Price

The predetermined price of an underlying asset is called as strike price.

3. ITÔ LEMMA

Statement Let X_t, t ε R_+, be an Itô process X : Ω x $R_+ \rightarrow$ R and $f := C^2\ (R\ x\ R_+\ x\ R_+)$. Then, the stochastic process

$f_t \coloneqq f(X_t\ , t)$is also an Itô process which satisfies

$$df_t = \left(\frac{\partial f}{\partial t} + a\ \frac{\partial f}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial t^2}\right) dt + \frac{\partial f}{\partial x} dW_t \qquad (1)$$

Proof: By Taylor's series the expansion of $f(X_{t+\Delta t}, t+\Delta t)$about $(X_t\ , t)$is given as follows

$$f(X_{t+\Delta t}, t+\Delta t)$$
$$= f(X_t,t) + \frac{\partial f}{\partial t}(X_t,t)(\Delta t) + \frac{\partial f}{\partial x}(X_t,t)(X_{t+\Delta t} - X_t) + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(X_t,t)(\Delta t)^2$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(X_t,t)(X_{t+\Delta t}, t - X_t)^2 + \frac{\partial^2 f}{\partial x \partial t}(X_t,t)(\Delta t)(X_{t+\Delta t} - X_t) + O(\Delta t)^2$$
$$+ O(\Delta t)(X_{t+\Delta t} - X_t)^2 + O((X_{t+\Delta t} - X_t)^3)$$

Taking, limit as $\Delta t \to 0$ gives

$$df_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dX_t{}^2 + O((\mathrm{d}t)^2) + O(dt(\mathrm{d}X_t)^2 + O\ ((dX_t)^3) \qquad (2)$$

Consider $X_t$is anItô Process and $dW_t{}^2 = dt$, we get

$$dX_t{}^2 = \ (adt + bdW_t)^2 = \ a^2(dt)^2 + 2abdtdW_t + \ b^2 dW_t{}^2$$
$$= \ b^2 \ + O((dt)^{3/2}) \qquad (3)$$

From stochastic differential equation, we get

$$df_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(adt + bdW_t) + \frac{1}{2}b^2\ \frac{\partial^2 f}{\partial t^2}dt$$

$$df_t = (\frac{\partial f}{\partial t} + a\ \frac{\partial f}{\partial x} + \frac{1}{2}b^2\ \frac{\partial^2 f}{\partial x^2})dt + b\frac{\partial f}{\partial x}dW_t (4)$$

where W_t is a Wiener process.

4. BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION.

Consider a general option values V(S, t). Therefore, from Taylor's theorem we have the following series expansion for V(S, t):

$$\delta V = V_s\delta S + V_t\delta t + \frac{1}{2!}V_{ss}\delta S^2 + \frac{1}{2!}V_{st}\delta S\delta t + \qquad \frac{1}{2!}V_{tt}\delta t^2 + ..(5)$$

We substitute $dS = vSdt + \ \sigma SdX$ in its discrete form, that is $\delta S = vS\delta t + \ \sigma S\delta X$ in equation (5), we get

$\delta V = V_s(vS\delta t + \sigma S\delta X) + V_t\delta t + \frac{1}{2!}V_{ss}(vS\delta t + \sigma S\delta X)^2 + \frac{1}{2!}V_{st}(vS\delta t + \sigma S\delta X)\delta t +$

$\frac{1}{2!}V_{tt}\delta t^2 + ..$ (6)

After, cancelling all insignificant terms in equation , we get

$\delta V \approx V_s(vS\delta t + \sigma S\delta X) + V_t\delta t + \frac{1}{2!}V_{ss}(vS\delta t + \sigma S\delta X)^2$(7)

Therefore, by taking the limits $\delta S \rightarrow 0, \delta X^2 \rightarrow \delta t\ as\ \delta t \rightarrow 0$ the above equation can be written in the following form:

$dV = V_s(vSdt + \sigma SdX) + V_t dt + \frac{1}{2!}V_{ss}(vSdt + \sigma SdX)^2$(8)

From equation (7), we have

$dS^s = (vSdt + \sigma SdX)^2 = (v^2S^2dt^2 + 2\sigma vS^2dXdt + \sigma^2S^2dX^2)$(9)

Therefore, by applying Itôs Lemma and assuming $dX^2 \rightarrow dt$ as $dt \rightarrow 0$, then from equation (9), we get

$dS^s \rightarrow \sigma^2S^2dt$ (10)

So we get

$dV = \frac{\partial V}{\partial S}(vSdt + \sigma SdX) + \frac{\partial V}{\partial t}dt + \frac{1}{2!}\frac{\partial^s V}{\partial S^2}(\sigma^2S^2dt)$(11)

Now rearranging the terms in above equation, we get

$$dV = \sigma S\frac{\partial V}{\partial S}dX + \left(vS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt(12)$$

This is the random walk process for V(S,t). By setting up a portfolio consisting of one option with value V(S,t) and a number -Δ of the underlying asset. The value of this portfolio will be

$\Pi = V - \Delta S$ (13)

Therefore, the change in the portfolio is

$d\Pi = dV - \Delta dS$(14)

Now, combining the above 3 equations, we get

$$d\Pi = \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)dX + \left(vS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^s V}{\partial S^2} + \frac{\partial V}{\partial t} - v\Delta S\right)dt \qquad (15)$$

To eliminate the main contribution of randomness, we choose

$$\Delta = \frac{\partial V}{\partial S} \qquad (16)$$

Now, theΔ is chosen such that the portfolio will be deterministic i.e. it is instantaneously risk free. The change in an instantaneously risk free portfolio should equal to the exponential growth of placing money in the bank.

Therefore, by using the value ofΔ and after simplification, we get

$$d\Pi = r\Pi \mathrm{dt} = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) dt \tag{17}$$

Finally, after using the value of Π and dividing it by dt, we obtain the equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{18}$$

This is the partial differential equation with variable coefficients is called the Black-Scholes equation for valuing an option with values V(S,t).

where

$V(S,t)$ – the price for an option

S – the current option price of the stock

r – the annualized risk-free interest rate, continuously compounded

t – the time in year generally use now t = 0, at expiry t = T

σ – volatility of an underlying asset.

This is second order partial differential equation in S - space and first order in time. Thus from its extensions and variants, it plays the major role in the option pricing theory. Thu, we have derived the Black-Scholes equation for the value of an option.

5. CONCLUSIONS

Black Scholes partial differential equation is highly important in financial engineering due to its accuracy and useful estimations of stock prices. This equation value option price by considering the factors like time period, risk-les interest rate and volatility of stock prices. It may also used in common stocks and bonds. We study the Ito's lemma and also derivation of Black-Scholes partial differential equation. This equation is widely used in financial world due to its efficiency and flexibility.

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